Dynamics

The Geometry of Behavior

Fourth Edition

Part 1

Periodic Behavior

with 342 illustrations

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Aerial Press

Aerial Press
P.O. Box 1360
Santa Cruz, California 95061

Design and Typography: Page & Curtis, Santa Cruz, California

Library of Congress Control Number 00-135270

Abraham, Ralph
Dynamics — the geometry of behavior / Ralph H. Abraham and
Christopher D. Shaw — 4th ed.

Part 1, Periodic Behavior

Four Volume Set
ISBN 0-942344-17-0

Part One
ISBN 0-942344-18-9

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Part 1: Periodic Behavior

Foreward vi
Preface vii
Part I Periodic Behavior 1
Dynamics Hall of Fame 3

1. Basic Concepts of Dynamics 13
1.1 State spaces 15
1.2 Dynamical systems 21
1.3 Special trajectories 30
1.4 Asymptotic approach to limit sets 36
1.5 Attractors, basins, and separatrices 42
1.6 Gradient systems 47

2. Classical Applications:
Limit Points in 2D from Newton to Rayleigh 53
2.1 Pendula 55
2.2 Buckling columns 65
2.3 Percussion instruments 71
2.4 Predators and prey 82

3. Vibrations:
Limit Cycles in 2D from Rayleigh to Rashevsky 87
3.1 Wind instruments 89
3.2 Bowed instruments 94
3.3 Radio transmitters 103
3.4 Biological morphogenesis 106

4. Forced Vibrations:
Limit Cycles in 3D from Rayleigh to Duffing 113
4.1 The ring model for forced springs 115
4.2 Forced linear springs 126
4.3 Forced hard springs 139
4.4 Harmonics 148

5. Compound Oscillations:
Invariant Tori in 3D from Huyghens to Hayashi 159
5.1 The torus model for two oscillators 161
5.2 The torus model for coupled oscillators 167
5.3 The ring model for forced oscillators 172
5.4 Braids: the dynamics of entrainment 176
5.5 Response curves for frequency changes 185
5.6 Forced electric oscillators 192

Conclusion 199

Forward

During the Renaissance, algebra was resumed from Near Eastern sources, and geometry from the Greek. Scholars of the time became familiar with classical mathematics. When calculus was born in 1665, the new ideas spread quickly through the intellectual circles of Europe. Our history shows the importance of the diffusion of these mathematical ideas, and their effects upon the subsequent development of the sciences and technology.

Today, there is a cultural resistance to mathematical ideas. Due to the widespread impression that mathematics is difficult to understand, or to a structural flaw in our educational system, or perhaps to other mechanisms, mathematics has become an esoteric subject. Intellectuals of all sorts now carry on their discourse in nearly total ignorance of mathematical ideas. We cannot help thinking that this is a critical situation, as we hold the view that mathematical ideas are essential for the future evolution of our society.

The absence of visual representations in the curriculum may be part of the problem, contributing to mathematical illiteracy and the math-avoidance reflex. This book is based on the idea that mathematical concepts may be communicated easily in a format that combines visual, verbal, and symbolic representations in tight coordination. It aims to attack math ignorance with an abundance of visual representations.

In sum, the purpose of this book is to encourage the diffusion of mathematical ideas by presenting them *visually*.

Preface

Dynamics is a field emerging somewhere between mathematics and the sciences. In our view, it is the most exciting event on the concept horizon for many years. The new concepts appearing in dynamics extend the conceptual power of our civilization and provide new understanding in many fields.

We discovered, while working together on the illustrations for a book in 1978,[1]* that we could explain mathematical ideas visually, within an easy and pleasant working partnership. In 1980, we wrote an expository article on dynamics and bifurcations,[2] using hand-animation to emulate the *dynamic picture technique* universally used by mathematicians in talking among themselves: a picture is drawn slowly, line by line, along with a spoken narrative - the dynamic picture and the narrative tightly coordinated.

Our efforts inevitably exploded into four volumes, now combined into this book. The dynamic picture technique, evolved through our work together, and in five years of computer graphic experience with the *Visual Math Project* at the University of California at Santa Cruz, is the basis of this work. The majority of the book is devoted to visual representations, in which four colors are used according to a strict code.

Math symbols have been kept to a minimum. In fact, they are almost completely suppressed. Our purpose is to make the book work for readers who are not practiced in symbolic representations. We rely exclusively on visual representations, with brief verbal explanations. Some formulas are shown with the applications, as part of the graphics, but are not essential. However, *this strategy is exclusively pedagogic*. We do not want anyone to think that we consider

**Footnotes refer to the Notes, which follow the appendix* symbolic representations unimportant in mathematics. On the contrary, this field evolved primarily in the symbolic realm throughout the classical period. Even now, a full understanding of our subject demands a full measure of formulas, logical expressions, and technical intricacies from all branches of mathematics. A brief introduction to these is included in the Appendix.

We have created this book as a short-cut to the research frontier of dynamical systems: theory, experiments, and applications. It is our goal - we know we may fail to reach it - to provide any interested person with an acquaintance with the basic concepts:

- state spaces: manifolds - geometric models for the virtual states of a system
- attractors: static, periodic, and chaotic - geometric models for its local asymptotic behavior
- separatrices: repellors, saddles, insets, tangles - defining the boundaries of regions (basins) dominated by different behaviors (attractors), and characterizing the global behavior of a system
- bifurcations: subtle and catastrophic - geometric models for the controlled change of one system into another.

The ideas included are selected from the literature of dynamics: Part One, "Periodic Behavior," covers the classical period from 1600 to 1950. Part Two, "Chaotic Behavior," is devoted to recent developments, 1950 to the present, on the chaotic behavior observed in experiments. Part Three, "Global Behavior," describes the concept of structural stability, discovered in 1937, and the important generic properties discovered since 1959, relating to the tangled insets and outsets of a dynamical system. These are fundamental to Part Four, "Bifurcation Behavior." In fact, the presentation in Part Four of an atlas of bifurcations in dynamical schemes with one control parameter was the original and primary goal of this whole book, and all of the topics in the first three parts have been selected for their importance to the understanding of these bifurcations.

*Footnotes refer to the Notes, which follow the Appendix.

For we regard the *response diagram*, a molecular arrangement of the atomic bifurcation events described here, as the most useful dynamical model available to a scientist.

We assume nothing in the way of prior mathematical training, beyond vectors in three dimensions, and complex numbers. Nevertheless, it will be tough going without a basic understanding of the simplest concepts of calculus.

Our first attempt at the pictorial style used here evolved in the first draft of *Dynamics: A Visual Introduction*, during the summer of 1980. Our next effort, the preliminary draft of Part Two of this book, was circulated among friends in the summer of 1981. Extensive feedback from them has been very influential in the evolution of this volume, and we are grateful to them:

Fred Abraham	George Francis
Jerry Marsden	Rob Shaw
Ethan Akin	Alan Garfinkel
Nelson Max	Mike Shub
Michael Arbib	John Guckenheimer
Jim McGill	Steve Smale
Jim Crutchfield	Moe Hirsch
Kent Morrison	Joel Smoller
Larry Cuba	Phil Holmes
Charles Musès	Jim Swift
Richard Cushman	Dan Joseph
Norman Packard	Bob Williams
Larry Domash	Jean-Michel Kantor
Tim Poston	Art Winfree
Jean-Pierre Eckman	Bob Lansdon
Otto Rössler	Marianne Wolpert
Len Fellman	Arnold Mandell
Lee Rudolph	Gene Yates
Katie Scott	Chris Zeeman

We are especially grateful to Tim Poston and Fred

Abraham for their careful reading of the manuscript; to the Dynamics Guild (J. Crutchfield, D. Farmer, N. Packard, and R. Shaw) for their computer plots used in many places in this book; to Richard Cushman for history lessons; to Page/Curtis for design and production; to Laura Logan of Aerial Press for her expert assistance in the production process; to Diane Rigoli for her splendid final drawings based on our rough sketches for Part Four; and to Rob Shaw for providing photos for Section 16.3 and computer plots for Section 17.3. The generosity and goodwill of many dynamicists has been crucial in the preparation of this book; we thank them all. We are grateful to Tom Jones, André Leroi-Gourhan, Preston James, Goeffrey Martin, and their publishers for permitting the reproduction of their illustrations. Finally, it is a pleasure to thank the National Science Foundation for financial support.

Ralph H. Abraham
Christopher D. Shaw
Santa Cruz, California
October, 1991

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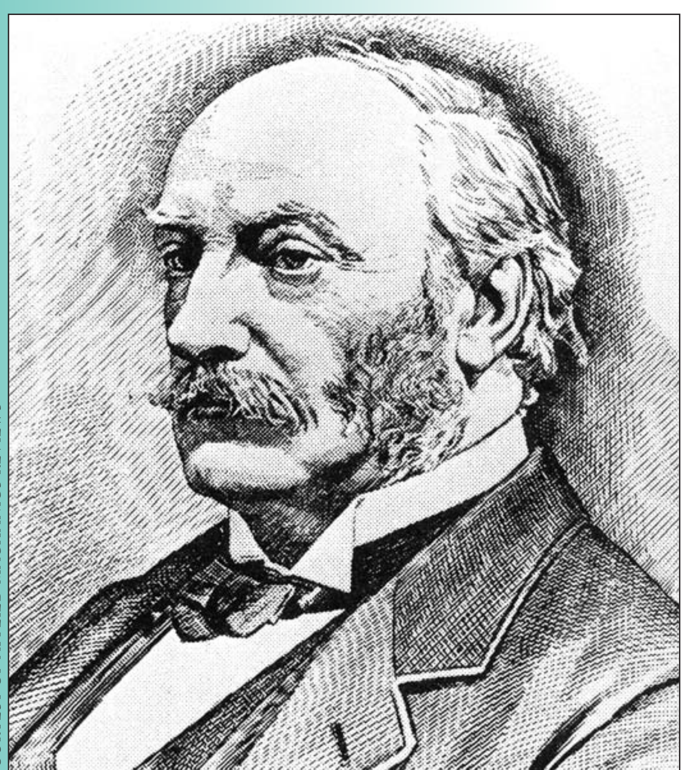
Dedicated to Lord Rayleigh



COURTESY OF APPLIED MACHANICS REVIEWS

Dynamics Hall of Fame

Dynamics has evolved into three disciplines: applied, mathematical, and experimental. Applied dynamics is the oldest. Originally regarded as a branch of natural philosophy, or physics, it goes back to Galileo at least. it deals with the concept of change, rate of change, rate of rate of change, and so on, as they occur in natural phenomena. We take these concepts for granted, but they emerged into our consciousness only in the fourteenth century.[1]

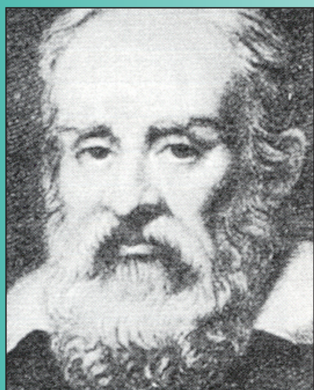
Mathematical dynamics began with Newton and has become a large and active branch of pure mathematics. This includes the theory of ordinary differential equations, now a classical subject. But since Poincaré, the newer methods of topology and geometry have dominated the field.

Experimental dynamics is an increasingly important branch of the subject. Founded by Galileo, it showed little activity until Rayleigh, Duffing, and Van der Pol. Experimental techniques have been revolutionized with each new development of technology. Analog and digital computers are now accelerating the advance of the research frontier, making experimental work more significant than ever.

This chapter presents a few words of description for some of the leading figures of the history of dynamics. Their positions in a two-dimensional tableau - date versus specialty (applied, mathematical, or experimental dynamics) - are shown in Table 1.1. Those included are not more important than numerous others, but limitations of space and knowledge prevent us from giving a more complete museum here.

Table 1.1
The History of Dynamics

Date	Applied Dynamics	Mathematical Dynamics	Experimental Dynamics
1600	Kepler		Galileo
1650			
	Huyghens	Newton	
		Leibniz	
1700			
		Euler	
1750			
		Lagrange	
1800			
1850			
	Helmholtz		
	Rayleigh	Poincaré	Rayleigh
		Lie	
		Liapounov	
1900			
			Duffing
	Lotka	Birkhoff	Van der Pol
	Volterra	Andronov	
	Rashevsky	Cartwright	Hayashi
1950			

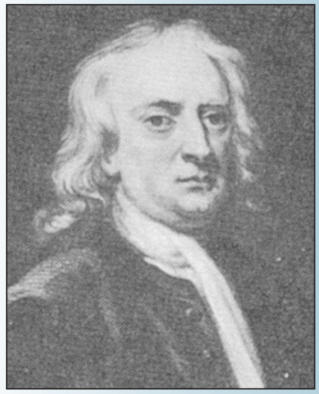


Galileo Galijei, 1564-1642. One of the first to deal thoroughly with the concept of acceleration, Galileo founded dynamics as a branch of natural philosophy. The close interplay of theory and experiment, characteristic of this subject, was founded by him.

Photo courtesy of DJ Struik, A Concise History of Mathematics, Dover Publications, New York (1948)

Johannes Kepler, 1571-1630. The outstanding and original exponent of applied dynamics. Kepler made use of extensive interaction between theory and observation to understand the planetary motions.

Courtesy of Kepler, Gesammelte Werke, Beck, München (1960)



Isaac Newton, 1642-1727. Mathematical dynamics, as well as the calculus on which it is based, was founded by Newton at age 23. Applications and experiments were basic to his ideas, which were dominated by the doctrine of determinism. His methods were geometric.

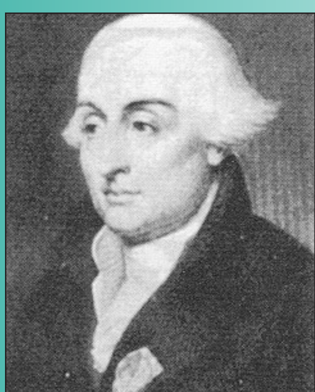
Photo courtesy of the Trustees of the British Museum

Gottfried Wilhelm Leibniz, 1646-1716. The concepts of calculus, mathematical dynamics, and their implications for natural philosophy, occurred independently to Leibniz. His methods were more symbolic than geometric.

Photo courtesy of the Trustees of the British Museum

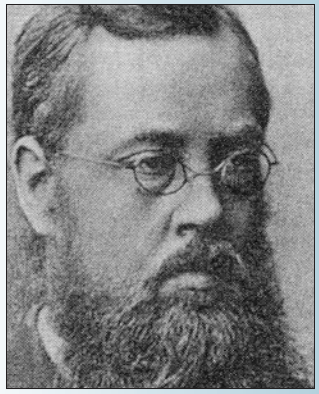
Leonhard Euler, 1707-1783. Primarily known for his voluminous contributions to algebra, Euler developed the techniques of analysis which were to dominate mathematical dynamics throughout its classical period.

Photo courtesy of E. T. Bell, Men of Mathematics, Simon and Schuster, New York (1937)



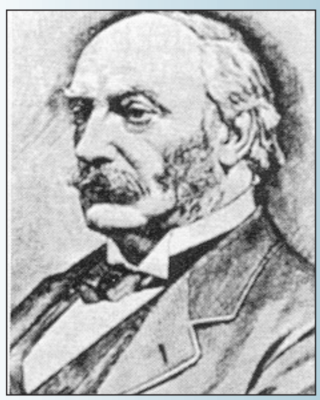
Joseph-Louis Lagrange, 1736-1813. A disciple of Euler, Lagrange developed the analytical method to extremes, and boasted that his definitive text on the subject contained not a single illustration.

Photo courtesy of the Bibliothéque Nationale, Paris, France



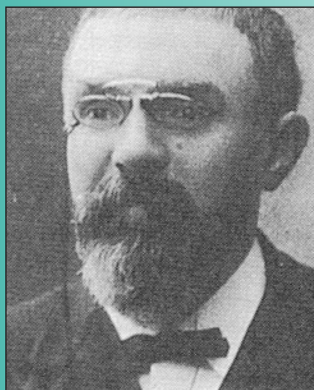
Marius Sophus Lie, 1842-1899. In combining the ideas of symmetry and dynamics, Lie built the foundations for a far-reaching extension of dynamics, the theory of groups of transformations.

Photo courtesy of Minkowski, H., Briefe an David Hilbhert, Mit Beiträgen und herausgegeben von L. Rüdenberg, H. Zassenbaus; Springer-Verlag, Heidelberg (1973)



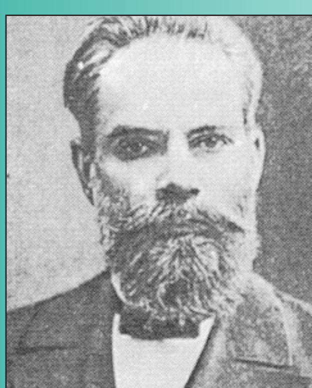
John William Strutt, Baron Rayleigh, 1842-1919. In a career of exceptional length and breadth, spanning applied mathematics, physics, and chemistry, Rayleigh dwelled at length on acoustical physics. In this context, he revived the experimental tradition of Galileo in dynamics, laying the foundations for the theory of nonlinear oscillations. His text on acoustics, published in 1877, remains to this day the best account of this subject.

Photo courtesy of Applied Mecb. Rev. 26 (1973)



Jules Henri Poincaré, 1854-1912. Known for his contributions to many branches of pure mathematics, Poincaré devoted the majority of his efforts to mathematical dynamics. Among the first to accept the fact that the classical analytical methods of Euler and Lagrange had serious limitations, he revived geometrical methods. The results were revolutionary for dynamics, and gave birth to topology and global analysis as well. These branches of pure mathematics are very active yet.

Photo courtesy of the Library of Congress, Washington, D. C.



Aleksandr Mikhailovich Liapounov, 1857-1918. Another pioneer of geometric methods in mathematical dynamics, Liapounov contributed basic ideas of stability.

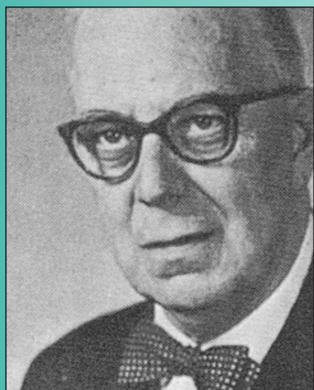
Photo courtesy of Akademija Nauk, SSR (1954)

Georg Duffing, 1861-1944. A serious experimentalist, Duffing studied mechanical devices to discover geometric properties of dynamical systems. The theory of oscillations was his explicit goal.

Photo courtesy of Mrs. Monika Murasch and Prof. Dr.- Ing. R. Gasch, Berlin.

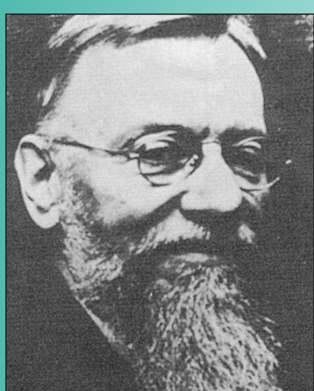
George David Birkhoff, 1884-1944. The first dynamicist in the New World, Birkhoff picked up where Poincaré left off. Although a geometer at heart, he discovered new symbolic methods. He saw beyond the theory of oscillations, created a rigorous theory of ergodic behavior, and foresaw dynamical models for chaos.

Photo courtesy of G. D. Birkhoff, Collected Mathematical Papers, American Mathematical Society, New York (1950)



Balthasar van der Pol, 1889-1959. The first radio transmitter became, in the hands of this outstanding experimentalist, a high-speed laboratory of dynamics. Many of the basic ideas of modern experimental dynamics came out of this laboratory.

Photo courtesy of Balthasar van der Pol, Selected Scientific Papers, Vol. 1, H. Bremmer and C. J. Boukamp (eds.), North Holland, Amsterdam (1960)



Nicholas Rashevsky, 1899-1972. From antiquity until the 1920's, applied dynamics meant physics. At last, the important applications to the biological and social sciences came into view, in the visionary minds of the general scientists Lotka, Volterra, and Rashevsky.

Photo courtesy of Bull. Math. Biophys. 34 (1972)

Mary Lucy Cartwright, 1900- . Dame Cartwright, together with J. E. Littlewood, revived dynamics in England, during World War II. Inspired by the work of Van der Pol, they obtained important results on the ultraharmonics of forced electronic oscillations, using analytical and topological methods.

Photo courtesy of Math. Gazette 36 (1952)

Chihiro Hayashi, 1911-1986. The experiments of dynamicists were restricted to a few simple systems (Duffing's system, Van der Pol's system, etc.) until the appearance of the general purpose analog computer. One of the creators of this type of machine, and the first to fully exploit one as a laboratory of dynamics, Hayashi contributed much to our knowledge of oscillations.

Photo courtesy of Ch. Hayashi,
Selected Papers on Nonlinear Oscillations, Kyoto (1975)